A note on symmetric functions

1996 J. Phys. A: Math. Gen. 29 L639

(http://iopscience.iop.org/0305-4470/29/24/006)

J. Phys. A: Math. Gen. **29** (1996) L639–L642. Printed in the UK

LETTER TO THE EDITOR

A note on symmetric functions

Allison Plant

Department of Mathematics, University of Tasmania, GPO Box 252 C, Hobart, Tasmania 7001, Australia

Received 10 September 1996

Abstract. We give the power-sum symmetric functions in terms of the Q-functions with spin character coefficients.

1. Introduction

The symmetric group $\mathcal{S}_n$ has a two-value representation (for $n \geqslant 4$) known as the spin representation. The spin representation matrices form a new group, the spin group Γ_n of order $2.n!$. Corresponding to each element of $\mathcal{S}_n$ there are two elements of Γ_n. We disregard those same characters of Γ_n and the remaining characters of Γ_n are called *spin characters*.

Schur [7] introduced the spin characters in 1911 as 'characters of the second kind' in his well known paper on the projective representations of the symmetric group. Since then much work has been done on the development of the theory of symmetric functions in conjunction with the ordinary representations. Similar results for the projective representations have not come as easily. A recent monograph by Hoffman and Humphreys [1] is devoted entirely to projective representations of the symmetric functions and gives a self-contained account of the algebraic theory of Q-functions. Spin characters play a key role in the developing area of Q-functions. These types of symmetric functions are useful in areas of physics such as nonlinear wave theory, where spin characters provide the coefficients for the solutions of certain types of soliton equations [6].

2. Preliminaries

A *partition* λ is a finite (or infinte) sequence of non-negative integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$$

arranged in descending order

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_d \geqslant 0.$$

The components λ_i of the partition λ are called *parts*, and the number of parts in a partition λ is called the *length* and is denoted $l(\lambda)$. The number of occurrences of a part λ_i in a partition λ is called the *multiplicity of* λ_i *in* λ and is denoted $m_{\lambda_i}(\lambda)$.

We will denote by $\mathbb{Z}$ the integers and by $\mathbb{Q}$ the rationals.

The symmetric group $\mathcal{S}_n$ acts on the ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ of polynomials with integer coefficients by permuting the variables. An element $f(x)$ from the ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$

of polynomials is said to be *symmetric* if it is invariant under the action of the symmetric group $\mathcal{S}_n$. The set of symmetric polynomials forms a sub-ring

$$\Lambda_n = \; Z[x_1, x_2, \ldots, x_n]^{\mathcal{S}_n}$$

of the ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ of polynomials.

This ring λ_n of symmetric polynomials is graded. That is

$$\Lambda_n = \bigoplus_{k \geqslant 0} \Lambda_n^k$$

where Λ_k^n is the group of the homogeneous symmetric functions of degree k in n variables, including the zero polynomial. In the theory of symmetric functions it is often more convenient to work in infinitely many variables. To this end, we define the group Λ^k by the inverse limit:

$$\Lambda^k = \lim_{\leftarrow n} \Lambda_n^k.$$

The *graded ring of symmetric functions* is defined as

$$\Lambda = \bigoplus_{k \geqslant 0} \Lambda^k.$$

There are various classical symmetric functions: elementary symmetric functions; monomial symmetric functions; complete symmetric functions; power-sum symmetric functions; Schur symmetric functions. Here we are only concerned with the power-sum symmetric functions.

The *rth power-sum symmetric function* is given by

$$p_r(x) = \sum_{i=1}^{\infty} x_i^r \qquad (r \geqslant 1)$$

$$p_0(x) = 1.$$

The power-sum symmetric function is multiplicative. That is

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_d}$$

for any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$.

Proposition 1 [4]. The power-sum symmetric functions form a $\mathbb{Q}$-basis for the graded ring of symmetric functions. That is, the power-sum symmetric functions are algebraically independent over $\mathbb{Q}$ and

$$\Lambda_{\mathbb{Q}} = \mathbb{Q}[p_1, p_2, \ldots].$$

We now turn our attention to Hall–Littlewood polynomials, first defined (indirectly) by Philip Hall in terms of the Hall algebra, and then directly by Dudley Littlewood in his paper 'On certain symmetric functions' which appeared in 1961. These functions are defined in the ring $\Lambda[t]$ of symmetric functions with coefficients in $\mathbb{Z}[t]$.

The *Hall–Littlewood polynomials* are defined by

$$P_\lambda(x_1, \ldots, x_n; t) = \sum_{\omega \in \mathcal{S}_n/\mathcal{S}_n^\lambda} \omega\left(x_1^{\lambda_1}, \ldots, x_n^{\lambda_n} \prod_{\lambda_1 > \lambda_j} \frac{x_i - t x_j}{x_i - x_j} \right)$$

where $\mathcal{S}_n^\lambda$ is the subgroup of permutations $\omega \in \mathcal{S}_n$ such that $\lambda_{w(i)} = \lambda_i$. The Hall–Littlewood polynomials are symmetric in $x_1, \ldots, x_n$ with coefficients in $\mathbb{Z}[t]$.

Again we pass to the limit and define the *Hall–Littlewood functions* $P_\lambda(x; t)$ as the elements from the ring $\Lambda[t]$ whose image in $\Lambda_n[t]$ for each $n \geqslant l(\lambda)$ is $P_\lambda(x_1, \ldots, x_n; t)$. When $t = 0$ these symmetric functions correspond with Schur symmetric functions.

Proposition 2 [4]. The Hall–Littlewood functions $P(x;t)$ are algebraically independent over $\mathbb{Z}[t]$ and form a $\mathbb{Z}[t]$-basis for the ring $\Lambda[t]$.

The *Q-functions* $Q_\lambda(x;t)$ are defined as scalar multiples of Hall–Littlewood functions as follows:

$$Q_\lambda(x;t) = b_\lambda(t)P_\lambda(x;t)$$

where

$$b_\lambda(t) = \prod_{i\geqslant 1}\varphi_{m_i(\lambda)}(t)$$

and

$$\varphi_r(t) = (1-t)(1-t^2),\ldots(1-t^n).$$

We are only concerned with the case when $t = -1$, as this gives rise to spin characters [1]. It is easy to see that in this case $\varphi_r(-1)$ is non-vanishing only for $r = 1$. That is, for the Q-functions $Q_\lambda(x;-1)$ we are only dealing with partitions λ with distinct parts.

We now define *Hall–Littlewood complete symmetric functions* q_r in terms of Hall–Littlewood functions:

$$q_r(x;t) = (1-t)P_r(x;t) \qquad (r \geqslant 1)$$

$$q_0(x;t) = 1.$$

Finally, for any partition λ we define

$$z_\lambda(t) = z_\lambda\prod_{i\geqslant 1}(1-t^{\lambda_i})^{-1}$$

where

$$z_\lambda = \prod_{i\geqslant 1} i_{m_i}.m_i!.$$

3. Results

MacDonald [4] gives the relationship between the Q-functions and the power-sum symmetric functions:

$$Q_\lambda(x;t) = \sum_\rho z_\rho^{-1}(t)X_\rho^\lambda(t)p_\rho(x) \tag{1}$$

where $X_\rho^\lambda(t)$ is the transition matrix between the power-sum symmetric functions $p_\rho(x)$ and the P-functions $P_\lambda(x;t)$.

We wish to express power-sum symmetric functions in terms of Q-functions with spin character coefficients.

From Morris [5] and Hoffman and Humphreys [1] we have

$$Q_\lambda(x;t) = \sum_\rho 2^{\frac{1}{2}(l(\lambda)+l(\rho)+\epsilon)}\frac{h_\rho}{h}\zeta_\rho^\lambda S_\rho \tag{2}$$

where h_ρ denotes the order of the class of ρ, h denotes the order of the irreducible representation Γ^λ, the S_ρ are the transitions which generate the symmetric group, ζ_ρ^λ is the spin character of the class ρ in the irreducible representation Γ^λ, and $\epsilon = 0$ or 1 appropriately (that is $\epsilon = 1$ when $l(\lambda) + l(\rho)$ is odd).

It is well known that the class ρ is an odd part partition and the irreducible representation Γ^λ is in a distinct part partition λ.

Lemma 3.

$$Q_\lambda(x;-1) = \sum_\rho z_\rho^{-1}(-1)2^{\frac{l(\lambda)-l(\rho)+\epsilon}{2}}\zeta_\rho^\lambda p_\rho(x) \tag{3}$$

Proof. Morris [5] states a formula for q_r first given by Schur [7]:

$$q_r(x;-1) = \sum_\rho \frac{h_\rho}{h} 2^{l(\rho)} S_\rho(x). \tag{4}$$

Stembridge [8] shows that

$$q_r(x;t) = \sum_\rho z_\rho^{-1}(t) p_\rho(x).$$

We are interested in the case when ρ is an odd part partition and $t=-1$ and in this case we have

$$z_\rho^{-1}(-1) = z_\rho^{-1}2^{l(\rho)}.$$

Hence

$$q_r(x;-1) = \sum_\rho z_\rho^{-1}2^{l(\rho)} p_\rho.$$

It now follows that

$$\frac{h_\rho}{h} S_\rho = z_\rho^{-1} p_\rho.$$

Substituting into equation (2) completes the proof. □

Corollary 4.

$$X_\rho^\lambda(-1) = 2^{\frac{l(\lambda)-l(\rho)+\epsilon}{2}}\zeta_\rho^\lambda. \tag{5}$$

Proof. This result follows directly from equation (1) and lemma 3. □

With this corollary, some algebraic manipulation and the observation that when $t=-1$ we have $b_\lambda = 2^{l(\lambda)}$ we come to the main result.

Theorem 5.

$$p_\rho(x) = \sum_\lambda 2^{-\frac{1}{2}(l(\rho)+l(\lambda)+\epsilon)}\zeta_\rho^\lambda Q_\lambda(x;-1).$$

References

[1] Hoffman P N and Humphreys J F 1992 *Projective Representations of the Symmetric Group* (Oxford: Clarendon)

[2] Józefiak T 1990 Schur Q-functions and applications, unpublished

[3] Littlewood D E 1950 *The Theory of Group Characters and Matrix Representations of Groups* (Oxford: Clarendon) 2nd edn

[4] MacDonald I G 1979 *Symmetric Functions and Hall Polynomials* (Oxford: Clarendon)

[5] Morris A O 1962 The spin representation of the symmetric group *Proc. London Math. Soc.* (3rd series) **XII** 55–76

[6] Plant A and Salam M A 1995 Schur's Q-functions and the BKP hierarchy *J. Phys. A: Math. Gen.* **28** L59–62

[7] Schur I 1911 Über die reelen Darstellung der symmetrischen und der alterneirenden Gruppe durch lineare Substitutionen *J.Reine Angew. Math.* **139** 155–250 (not directly accessed)

[8] Stembridge J R 1989 Shifted tableaux and the projective representation of symmetric groups *Adv. Math.* **74** 87–134